

Error Analysis for RMS Emittance Measurements

we, 11-18-87

LU-108

The intensity I data is stored in the computer as a matrix I_{ij} where the i subscript refers to the x position and the j subscript gives x' . The $x-x'$ plane is covered with a grid which has the axes divided up into constant intervals Δx and $\Delta x'$. Thus we can write instead of the integral for a moment of x and x'

$$\langle x^n x'^m \rangle = \frac{\sum_{i,j} I_{ij} x_i^n x_j'^m \Delta x \Delta x'}{\sum_{i,j} I_{ij} \Delta x \Delta x'} = \frac{\sum I_{ij} x_i^n x_j'^m}{\sum I_{ij}}$$

The starting point is the uncertainties in a single measurement of x or x' which are respectively Δx and $\Delta x'$. These must be determined by the experimentalist. Let us find the errors in the means of x and x' . We have

$$\langle x \rangle \equiv \bar{x} = \frac{\sum I_{ij} x_i}{\sum I_{ij}}, \quad \frac{\partial \bar{x}}{\partial x_k} = \frac{\sum I_{kj}}{\sum I_{ij}} = \frac{I_k}{\sum I_{ij}}$$

where we have defined $I_k = \sum_j I_{kj}$ = the projection of $I(x, x')$ onto the (x, I) plane.

Our estimate for the error of the mean \bar{x} is then

$$\sigma_{\bar{x}}^2 = \sum_k \left(\frac{\partial \bar{x}}{\partial x_k} \right)^2 \Delta x_k^2 = \Delta x^2 \frac{\sum I_k^2}{(\sum I_{ij})^2}$$

We assume that all the Δx_k are equal. Analogously for \bar{x}' we have

$$\sigma_{\bar{x}'}^2 = \Delta x'^2 \cdot \frac{\sum I_j^2}{(\sum I_{ij})^2}$$

where $I_j = \sum_i I_{ij}$

is the projection of the intensity contour onto $x'-I$ space.

We can now determine the error in the elements of the covariance matrix $\sigma_{xx}, \sigma_{x'x'}, \sigma_{x'x}$.

$$\sigma_{xx} = \bar{x^2} - \bar{x}^2$$

$$\text{We need the error in } \bar{x^2} = \frac{\sum I_{ij} x_i^2}{\sum I_{ij}} = \frac{\sum I_i x_i^2}{\sum I_{ij}}$$

where we have again summed over j .

$$\frac{\partial \bar{x^2}}{\partial x_i} = \frac{2 I_i x_i}{\sum I_{ij}} \Rightarrow \sigma_{\bar{x^2}}^2 = \sum_i \left(\frac{\partial \bar{x^2}}{\partial x_i} \right)^2 \Delta x_i^2$$

$$\sigma_{\bar{x^2}}^2 = \frac{4 \Delta x^2}{(\sum I_{ij})^2} \cdot \sum (I_i x_i)^2 ; \quad \sigma_{\bar{x}} = \frac{2 \Delta x}{\sum I_{ij}} \sqrt{\sum (I_i x_i)^2}$$

$$\text{Now } \frac{\partial \sigma_{xx}}{\partial \bar{x^2}} = 1, \quad \frac{\partial \sigma_{xx}}{\partial \bar{x}} = -2 \bar{x}$$

$$\therefore \Delta_{xx}^2 = \sigma_{\bar{x^2}} + 4 \bar{x^2} \sigma_{\bar{x}}^2 = \left(\frac{2 \Delta x}{\sum I_{ij}} \right)^2 \left[\sum (I_i x_i)^2 + \frac{(\sum I_{ij} x_i)^2 \cdot \sum (I_i)^2}{(\sum I_{ij})^2} \right]$$

Thus the error of σ_{xx} is

$$\Delta_{xx} = \frac{2 \Delta x}{\sum I_{ij}} \left[\sum (I_i x_i)^2 + \frac{(\sum I_i x_i)^2 \cdot \sum (I_i)^2}{(\sum I_{ij})^2} \right]^{1/2}.$$

Similarly

$$\Delta_{x'x'} = \frac{2 \Delta x'}{\sum I_{ij}} \left[\sum (I_j x'_j)^2 + \frac{(\sum I_j x'_j)^2 \cdot \sum (I_j)^2}{(\sum I_{ij})^2} \right]^{1/2}.$$

Next we determine the error in $\bar{v}_{xx'}$. $\sigma_{xx'} = \bar{xx'} - \bar{x} \cdot \bar{x}'$

$$\bar{xx'} = \frac{\sum I_{ij} x_i x'_j}{\sum I_{ij}}, \quad \frac{\partial \bar{xx'}}{\partial x_i} = \frac{\sum I_{ij} x'_j}{\sum I_{ij}} = \frac{I_j x'_j}{\sum I_{ij}},$$

$$\frac{\partial \bar{xx'}}{\partial x'_j} = \frac{I_i x_i}{\sum I_{ij}}, \quad \text{thus}$$

$$\begin{aligned} \sigma_{xx'}^2 &= (\text{error})^2 \text{ of } \bar{xx'} \neq \bar{v}_{xx'} \\ &= \frac{1}{(\sum I_{ij})^2} \cdot \left[\Delta x^2 \cdot \sum (I_i x_i)^2 + \Delta x'^2 \cdot \sum (I_j x'_j)^2 \right] \end{aligned}$$

$$\begin{aligned} \text{Now } \Delta_{xx'}^2 &= \left(\frac{\partial \bar{v}_{xx'}}{\partial (\bar{xx'})} \right)^2 \cdot \sigma_{\bar{xx'}}^2 + \left(\frac{\partial \bar{v}_{xx'}}{\partial \bar{x}} \right)^2 \cdot \sigma_{\bar{x}}^2 + \left(\frac{\partial \bar{v}_{xx'}}{\partial \bar{x}'} \right)^2 \cdot \sigma_{\bar{x}'}^2 \\ &= \sigma_{\bar{xx'}}^2 + \bar{x}'^2 \cdot \sigma_{\bar{x}}^2 + \bar{x}^2 \cdot \sigma_{\bar{x}'}^2 \\ &= (\sum I_{ij})^{-2} \cdot \left[\Delta x^2 \cdot \left(\sum (I_j x'_j)^2 + \bar{x}'^2 \cdot \sum (I_j)^2 \right) + \right. \\ &\quad \left. \Delta x'^2 \cdot \left(\sum (I_i x_i)^2 + \bar{x}^2 \sum (I_i)^2 \right) \right]. \end{aligned}$$

Now let $a = \sum I_{ij}$, $b = \sum I_i^2$, $c = \sum I_j^2$, $d = \sum (I_i x_i)^2$,
 $e = \sum (I_j x'_j)^2$. We can summarize our formulas

as

$$\sigma_{\bar{x}} = \frac{\Delta x \cdot \sqrt{b}}{a}, \quad \sigma_{\bar{x}'} = \frac{\Delta x' \cdot \sqrt{c}}{a},$$

$$\Delta_{xx} = \frac{2\Delta x}{a} \sqrt{d + b \cdot \bar{x}^2}, \quad \Delta_{x'x'} = \frac{2\Delta x'}{a} \sqrt{e + c \cdot \bar{x}'^2},$$

and

$$\Delta_{xx'} = \frac{1}{a} \sqrt{\Delta x^2 \cdot (e + \bar{x}'^2 \cdot b) + \Delta x'^2 \cdot (d + c \cdot \bar{x}^2)}.$$

These last three errors will be used to determine the error in ϵ and α, β , and γ . Recall that

$$\epsilon = 4 \sqrt{\sigma_{xx} \cdot \sigma_{x'x'} - \sigma_{xx'}^2},$$

We easily find the error in $\epsilon, \delta\epsilon$ to be

$$\delta\epsilon = \frac{8}{\epsilon} \sqrt{\Delta_{xx}^2 \cdot \sigma_{x'x'}^2 + \Delta_{x'x'}^2 \cdot \sigma_{xx}^2 + 4 \Delta_{xx'}^2 \cdot \sigma_{xx'}^2},$$

from $\beta = \frac{4\sigma_{xx}}{\epsilon}$, $\gamma = \frac{4\sigma_{x'x'}}{\epsilon}$, $\alpha = -\frac{4\sigma_{xx'}}{\epsilon}$ we find

$$\delta\beta = \frac{4}{\epsilon} \left[\Delta_{xx}^2 + \sigma_{xx}^2 \left(\frac{\delta\epsilon}{\epsilon} \right)^2 \right]^{1/2},$$

$$\delta\gamma = \frac{4}{\epsilon} \left[\Delta_{x'x'}^2 + \sigma_{x'x'}^2 \left(\frac{\delta\epsilon}{\epsilon} \right)^2 \right]^{1/2},$$

and

$$\delta\alpha = \frac{4}{\epsilon} \left[\Delta_{xx'}^2 + \sigma_{xx'}^2 \left(\frac{\delta\epsilon}{\epsilon} \right)^2 \right]^{1/2}.$$